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On Shifted Cardinal Interpolation by Gaussians and Multiquadrics

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A radial basis function approximation is a linear combination of translates of a fixed function $\varphi: \mathscr{R}^d \to \mathscr{R}$. Such functions possess many useful and interesting properties when the translates are integers and φ is radially symmetric. We study the closely related problem for which the fixed function is the shifted Gaussian $\varphi = G(\cdot - \alpha)$, where $G(x) = \exp(-\lambda \|x\|_2^2)$ and $\alpha \in \mathscr{R}^d$. Specifically, we exploit the theory of elliptic functions to establish the invertibility of the Toeplitz operator

$$(\varphi(\alpha + j - k))_{j, k \in \mathscr{Z}^d}$$

when α has no half-integer components; it is singular otherwise. This implies the existence of a *shifted Gaussian cardinal function*, that is, a linear combination χ of integer translates of the shifted Gaussian satisfying $\chi(j) = \delta_{0j}$. We also study shifted cardinal functions when the parameter λ tends to zero. In particular, we discover their uniform convergence to the sinc function when the shift vector α possesses no half-integer components. Our methods are based in part on similar results established by the first author when the basis function is the Hardy multiquadric. Several intriguing links with the theory of shifted B-spline cardinal interpolation are described in the finale.

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1. INTRODUCTION

A radial basis function approximant is a linear combination of translates of a fixed function, or some suitable limit of such approximants. Thus we consider

$$s(x) = \sum_{k \in \mathscr{Z}^d} a_k \varphi(x - b_k), \qquad x \in \mathscr{R}^d, \tag{1.1}$$

where $(b_k)_{k \in \mathscr{Z}^d}$ is some fixed set of distinct points, or *centres*, in $\mathscr{R}^d$, and $(a_k)_{k \in \mathscr{Z}^d}$ is a sequence of real numbers satisfying conditions ensuring (1.1) is meaningful; for example, we might require the scalar sequence to be finitely supported, or for the infinite series in (1.1) to be absolutely convergent at every point $x \in \mathscr{R}^d$. Such functions provide a flexible and useful approach to multivariate interpolation (see, for example, the survey articles [P, Bu3]). Much of the existing literature concentrates on the special case when the centres form an infinite grid and φ is radially symmetric; we refer the reader to the fundamental papers [Bu1, Bu2] of Buhmann on cardinal interpolation. Here we study the closely related problem of shifted Gaussian cardinal interpolation, which means that our typical approximant is

$$s(x) = \sum_{k \in \mathscr{Z}^d} a_k \varphi(x + \alpha - k), \qquad x \in \mathscr{R}^d, \tag{1.2}$$

where the *shift* α is a fixed vector in $\mathscr{R}^d$ and the function $\varphi(x) = \exp(-\lambda \, \|x\|_2^2)$ is a Gaussian. We shall also allow the (positive) parameter λ to vary. First, let us recall that the cardinal function χ_α for the shifted Gaussian must satisfy

$$\chi_\alpha(j) = \delta_{0j}, \qquad j \in \mathscr{Z}^d, \tag{1.3}$$

where

$$\chi_\alpha(x) = \sum_{k \in \mathscr{Z}^d} c_k(\alpha)\, \varphi(x + \alpha - k), \qquad x \in \mathscr{R}^d. \tag{1.4}$$

Our main finding is that such cardinal functions exist when the shift vector α has no *half-integer* components, the term half-integer connoting an element of $\frac{1}{2} + \mathscr{Z}$ in this paper, and we shall call such shifts *admissible*. The technique is founded on analysis of the non-Hermitian Toeplitz operator

$$(\varphi(\alpha + j - k))_{j, k \in \mathscr{Z}^d}$$

using its close links with the theory of Jacobian Theta functions; these links were also exploited by the first author in [B1]. This analysis is to be found in Section 2. In Section 3, we prove that admissibly shifted cardinal functions converge uniformly to the sinc function as the parameter λ tends to zero. Moreover, we also show that the admissibly shifted cardinal interpolants to a square-integrable function converge to this function in the mean-square sense if and only if it is band limited. These studies indicate that excellent accuracy can be attained when approximating band-limited functions by shifted Gaussians if the parameter λ is suitably small. We suspect that this is mostly responsible for the favourable results found in, say, the applications of Gaussian radial basis functions in neural net problems (see, for instance [BL]).

The methods employed in Section 3 are based in part on similar results established by the first author in [B2] when the basis function is the Hardy multiquadric $\varphi(x)=(\|x\|_2^2+c^2)^{1/2}$ and the parameter c tends to infinity; we discuss this connection in Section 4. Furthermore, our Gaussian researches shed some light on shifted multiquadric interpolation, and these implications are also outlined in Section 4. Finally, Section 5 describes the intriguing parallels between the theory of shifted B-spline cardinal interpolation and this paper.

2. SHIFTED GAUSSIANS, TOEPLITZ FORMS, AND THETA FUNCTIONS

Let λ be a positive constant, let $\varphi\colon \mathscr{R}\to\mathscr{R}$ be the Gaussian $\varphi(x)=\exp(-\lambda x^2)$, $x\in\mathscr{R}$, and define the shifted Gaussian $\varphi_\alpha(x)=\varphi(x+\alpha)$, where α is a real number. We consider the bi-infinite Toeplitz matrix

$$A(\alpha):=(\varphi_\alpha(j-k))_{j,\,k\in\mathscr{Z}}, \qquad \alpha\in\mathscr{R}, \tag{2.1}$$

as a linear operator on $l^2(\mathscr{Z})$. The classical theory of Toeplitz forms (see [W] or [GS]) studies $A(\alpha)$ via the *symbol function*

$$G_\alpha(\xi)\equiv G_\alpha(\xi,\lambda):=\sum_{k\in\mathscr{Z}}\varphi_\alpha(k)\exp(-ik\xi), \qquad \xi\in\mathscr{R}, \tag{2.2}$$

and we recall the well-known fact that $A(\alpha)\colon l^2(\mathscr{Z})\to l^2(\mathscr{Z})$ is invertible if and only if the symbol function does not vanish [W, Theorem 1]. Following [B1] we find that G_α is a multiple of the Theta function

$$\vartheta(z)\equiv\vartheta(z,q):=\sum_{k\in\mathscr{Z}}q^{k^2}z^k, \qquad z\in\mathscr{C}\setminus\{0\}, \quad \text{for } q\in\mathscr{C} \text{ and } |q|<1. \tag{2.3}$$

Specifically, some elementary algebraic manipulation reveals the identity

$$G_\alpha(\xi,\lambda)=q^{\alpha^2}\vartheta(q^{2\alpha}e^{-i\xi}) \qquad \text{where} \quad q=e^{-\lambda}. \tag{2.4}$$

Thus the invertibility of $A(\alpha)$ is determined by the zero structure of the associated Theta function. Therefore we present below some salient properties of Theta functions needed later in the paper.

LEMMA 2.1. *The Theta function enjoys the infinite product formula*

$$\vartheta(z) = T(q) \prod_{k=0}^{\infty} (1+q^{2k+1}z)(1+q^{2k+1}z^{-1}), \qquad z \in \mathscr{C} \setminus \{0\},$$

where

$$T(q) := \prod_{l=1}^{\infty} (1-q^{2l}). \tag{2.5}$$

Proof. See [WW, Section 21.3; Be, Section 32]. ∎

COROLLARY 2.2. *The zeros of ϑ are given by $\{-q^l : l \in \mathscr{Z} \text{ odd}\}$.*

Proof. Equation (2.5) implies that $\vartheta(z)=0$ if and only if $1+q^{2k+1}z^{\pm 1}=0$ for some non-negative integer k. ∎

PROPOSITION 2.3. *The function $\{\xi \mapsto |\vartheta(q^{2\alpha}e^{-i\xi})|^2 : \xi \in \mathscr{R}\}$ is even, 2π-periodic, and decreases for $0 \leqslant \xi \leqslant \pi$.*

Proof. The function is evidently even and 2π-periodic. Furthermore, (2.5) yields the infinite product

$$\begin{aligned} |\vartheta(q^{2\alpha}e^{-i\xi})|^2 &= T(q)^2 \prod_{k=0}^{\infty} (1+q^{2k+1+2\alpha}e^{-i\xi})(1+q^{2k+1+2\alpha}e^{i\xi}) \\ &\quad \times (1+q^{2k+1-2\alpha}e^{i\xi})(1+q^{2k+1-2\alpha}e^{-i\xi}) \\ &= T(q)^2 \prod_{k=0}^{\infty} (1+2q^{2k+1+2\alpha}\cos\xi + q^{4k+2+4\alpha}) \\ &\quad \times (1+2q^{2k+1-2\alpha}\cos\xi + q^{4k+2-4\alpha}), \end{aligned} \tag{2.6}$$

and we see that each of the terms in the final product is decreasing for $0 \leqslant \xi \leqslant \pi$. ∎

PROPOSITION 2.4. *$\Re\vartheta(q^{2\alpha}w) \geqslant 0$ when $|w|=1$ and $\alpha \in [-1/2, 1/2]$, with equality if and only if $w=-1$ and $\alpha = \pm 1/2$.*

Proof. It suffices to prove this for $0 \leqslant \alpha \leqslant 1/2$ because of the equation $\vartheta(q^{-2\alpha}w) = \overline{\vartheta(q^{2\alpha}w)}$. Now the Theta function is a conformal mapping from the open annulus $\{z \in \mathscr{C} : q < |z| < 1\}$ onto the domain whose boundaries are the images under ϑ of the boundaries of the annulus. It is shown in [B1, Lemma 2.7] that ϑ maps the unit circle $\{z \in \mathscr{C} : |z| = 1\}$ onto the real interval $[\vartheta(-1), \vartheta(1)]$. Thus it suffices to prove that $\Re\vartheta(qw) \geqslant 0$ when $|w| = 1$. But (2.5) supplies the expression

$$\vartheta(qw) = T(q) \prod_{k=0}^{\infty} (1 + q^{2k+2}w)(1 + q^{2k}\bar{w}) = (1 + \bar{w})\, T(q) \left| \prod_{k=0}^{\infty} (1 + q^{2k}w) \right|^2,$$

whence $\Re\vartheta(qw) = (1 + \Re w)\, T(q)\, |\prod_{k=0}^{\infty} (1 + q^{2k}w)|^2 \geqslant 0$ with equality if and only if $w = -1$. ▮

All the results obtained hitherto will be used in the sequel. We commence with an invertibility theorem for $A(\alpha)$.

THEOREM 2.5. *The symbol function G_α, defined by* (2.2), *has no zeros unless α is a half-integer. Equivalently, the bi-infinite Toeplitz matrix $A(\alpha)$ of* (2.1) *is invertible on $l^2(\mathscr{Z})$ if and only if α is not a half-integer.*

Proof. Corollary 2.2 and (2.4) imply that the symbol function $G_\alpha(\xi, \lambda) = 0$ if and only if 2α is an odd integer and $e^{-i\xi} = -1$. The equivalence follows from [W, Theorem 1]. ▮

This result was found independently by R. A. Rahim [R], whose technique was quite different.

Given the invertibility of $A(\alpha)$ when $\alpha \notin 1/2 + \mathscr{Z}$, it is natural to consider the condition number $\operatorname{cond}_2 A(\alpha) := \|A(\alpha)\|\, \|A(\alpha)^{-1}\|$ for such α (the norm used here is the operator norm on $l^2(\mathscr{Z})$).

THEOREM 2.6. *Let $\alpha = \alpha_0 + l$, where $|\alpha_0| < 1/2$ and l is an integer. Then*

$$\operatorname{cond}_2 A(\alpha) = \frac{\vartheta(q^{2\alpha_0})}{\vartheta(-q^{2\alpha_0})}. \tag{2.7}$$

Proof. The spectrum of the bi-infinite Hermitian Toeplitz matrix $A(\alpha)^* A(\alpha)$ is given by the range of its symbol function $\{|G_\alpha(\xi)|^2 : -\pi \leqslant \xi \leqslant \pi\}$ (see [W, Theorem 1′]). Therefore $\|A(\alpha)^* A(\alpha)\| = \max\{|G_\alpha(\xi)|^2 : -\pi \leqslant \xi \leqslant \pi\}$, by [Bo, p. 175, Theorem 11(c)]. Furthermore, $\|A(\alpha)\|^2 = \|A(\alpha)^* A(\alpha)\|$ by [Bo, p. 157, Theorem 2], which yields $\|A(\alpha)\| = \max\{|G_\alpha(\xi)| : -\pi \leqslant \xi \leqslant \pi\}$. Similarly $\|A(\alpha)^{-1}\| = \max\{|G_\alpha(\xi)|^{-1} : -\pi \leqslant \xi \leqslant \pi\}$. Thus

$$\begin{aligned}
\operatorname{cond}_2 A(\alpha) &= \frac{\max\{|G_\alpha(\xi) : \xi \in [-\pi, \pi]\}}{\min\{|G_\alpha(\xi)| : \xi \in [-\pi, \pi]\}} \\
&= \frac{\max\{|G_{\alpha 0}(\xi)| : \xi \in [-\pi, \pi]\}}{\min\{|G_{\alpha_0}(\xi)| : \xi \in [-\pi, \pi]\}} \\
&= \frac{\max\{|\vartheta(q^{2\alpha_0} w)| : w \in \mathscr{C}, |w| = 1\}}{\min\{|\vartheta(q^{2\alpha_0} w)| : w \in \mathscr{C}, |w| = 1\}} \\
&= \left| \frac{\vartheta(q^{2\alpha_0})}{\vartheta(-q^{2\alpha_0})} \right|,
\end{aligned} \tag{2.8}$$

by (2.2), (2.4) and Proposition 2.3. Finally, Proposition 2.4 entails the non-negativity of $\vartheta(\pm q^{2\alpha_0})$. ▮

Equation (2.7) reflects the fact that $\operatorname{cond}_2 A(\alpha)$ is a 1-periodic function of the shift parameter α. Our next result shows that the condition number increases as $|\alpha|$ grows from zero to half.

THEOREM 2.7. *The function* $\{\alpha \mapsto \operatorname{cond}_2 A(\alpha) :\ -1/2 < \alpha < 1/2\}$ *is even, increasing on* $[0, 1/2)$, *and tends to infinity as* α *tends to* $1/2$.

Proof. Theorem 2.6 and Lemma 2.1 provide the relations

$$\begin{aligned}
\operatorname{cond}_2 A(\alpha) = \frac{\vartheta(q^{2\alpha})}{\vartheta(-q^{2\alpha})} &= \prod_{k=0}^{\infty} \frac{1 + q^{2k+1}(q^{2\alpha} + q^{-2\alpha}) + q^{4k+2}}{1 - q^{2k+1}(q^{2\alpha} + q^{-2\alpha}) + q^{4k+2}} \\
&= \prod_{k=0}^{\infty} \frac{1 + 2q^{2k+1} \cosh(2\lambda\alpha) + q^{4k+2}}{1 - 2q^{2k+1} \cosh(2\lambda\alpha) + q^{4k+2}},
\end{aligned} \tag{2.9}$$

and this last expression is clearly an even function of α. Further, each term in the numerator of the product increases for $0 \leqslant \alpha \leqslant 1/2$, whereas each term in the denominator decreases for $0 \leqslant \alpha \leqslant 1/2$. Finally, as α tends to $1/2$, the first term ($k = 0$) in the denominator tends to zero. ▮

It is interesting, but seemingly irrelevant, that the infinite product obtained in (2.9) is a multiple of the Jacobian elliptic function dn.

We now take up the analysis of the semi-infinite Toeplitz matrix

$$A_+(\alpha) := (\varphi_\alpha(j-k))_{j, k \geqslant 0}, \qquad \alpha \in \mathscr{R}, \tag{2.10}$$

viewed as a linear operator on $l^2(\mathscr{Z}_+)$; here $\varphi_\alpha(x) = e^{-\lambda(x+\alpha)^2}$ and $l^2(\mathscr{Z}_+)$ denotes the sequence space $\{(a_k)_{k \geqslant 0} : \sum_{k=0}^{\infty} |a_k|^2 < \infty\}$. The interaction between the symbol function and the semi-infinite Toeplitz operator is more subtle than in the bi-infinite case. We recall that a semi-infinite Toeplitz matrix associated with a continuous symbol σ is invertible on

$l^2(\mathscr{Z}_+)$ if and only if σ is nowhere zero and the curve $\{\sigma(t): -\pi \leqslant t \leqslant \pi\}$ does not wind about zero [W, Theorem 5].

PROPOSITION 2.8. *Suppose α is not a half-integer. Then $\{G_\alpha(\xi): -\pi \leqslant \xi \leqslant \pi\}$ does not wind about zero if and only if $|\alpha| < 1/2$.*

Proof. If $|\alpha| < 1/2$, then Proposition 2.4 yields $\Re\vartheta(q^{2\alpha}e^{-i\xi}) > 0$ for all $\xi \in [-\pi, \pi]$. Hence $\Re G_\alpha(\xi) = q^{\alpha^2}\Re\vartheta(q^{2\alpha}e^{-i\xi})$ is positive for all $\xi \in [-\pi, \pi]$; *a fortiori* $\{G_\alpha(\xi): -\pi \leqslant \xi \leqslant \pi\}$ does not wind about zero.

Conversely, let $\alpha = \alpha_0 + l$, where $|\alpha_0| < 1/2$ and $l \in \mathscr{Z} \setminus \{0\}$. Since $G_\alpha(\xi) = e^{il\xi}G_{\alpha_0}(\xi)$ by (2.2) and $\{G_{\alpha_0}(\xi): -\pi \leqslant \xi \leqslant \pi\}$ does not wind about the origin, we conclude that $\{G_\alpha(\xi): -\pi \leqslant \xi \leqslant \pi\}$ winds about zero exactly l times. ∎

THEOREM 2.9. *Suppose $\alpha \in \mathscr{R}$. The following are equivalent:*

(i) $|\alpha| < 1/2$;

(ii) *G_α is nowhere zero and the winding number of the curve $\{G_\alpha(\xi): -\pi \leqslant \xi \leqslant \pi\}$ about zero is zero.*

(iii) *The semi-infinite Toeplitz matrix $A_+(\alpha)$ defined in* (2.10) *is invertible on $l^2(\mathscr{Z}_+)$.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from Theorem 2.5 and Proposition 2.8.

(ii) $\Leftrightarrow$ (iii). This is [W, Theorem 5]. ∎

We close this section with multivariate extensions of Theorems 2.5 and 2.6. Let $\varphi^{(d)}(x) := \exp(-\lambda \|x\|_2^2)$, $x \in \mathscr{R}^d$, $\lambda > 0$, and let $\varphi_\alpha^{(d)}(x) := \varphi^{(d)}(x+\alpha)$, $\alpha \in \mathscr{R}^d$. Consider the multivariate Toeplitz matrix (see [BM])

$$A^{(d)}(\alpha) := (\varphi_\alpha^{(d)}(j-k))_{j,k \in \mathscr{Z}^d}, \qquad \alpha \in \mathscr{R}^d, \tag{2.11}$$

as a linear operator on $l^2(\mathscr{Z}^d)$. The symbol function $G_\alpha^{(d)}$ of $A^{(d)}(\alpha)$ is given by

$$G_\alpha^{(d)}(\xi) \equiv G_\alpha^{(d)}(\xi, \lambda) := \sum_{k \in \mathscr{Z}^d} \varphi_\alpha^{(d)}(k) \exp(-ik^T\xi), \qquad \xi \in \mathscr{R}^d. \tag{2.12}$$

Clearly $G_\alpha^{(d)}$ is a tensor product of univariate symbol functions, to wit

$$G_\alpha^{(d)}(\xi) = \prod_{k=1}^{d} G_{\alpha_k}(\xi_k), \qquad \xi = (\xi_1, ..., \xi_d), \qquad \alpha = (\alpha_1, ..., \alpha_d).$$

Consequently, Theorems 2.5 and 2.6 have multivariate analogues.

THEOREM 2.10. *The multivariate Toeplitz matrix* $A_\alpha^{(d)}$ *defined in* (2.11) *is invertible on* $l^2(\mathscr{Z}^d)$ *if and only if no co-ordinate of the vector shift* $\alpha=(\alpha_1, ..., \alpha_d)$ *is a half-integer.*

Proof. In view of (2.4) and Corollary 2.2, $G_\alpha^{(d)}(\xi)=0$, $\xi=(\xi_1, ..., \xi_d)$, if and only if $2\alpha_k$ is an odd integer and ξ_k is an odd integral multiple of π for some $k\in\{1, ..., d\}$. ∎

THEOREM 2.11. *Suppose* $\alpha=\alpha_0+l$, *where* $\alpha_0\in(-1/2, 1/2)^d$ *and* $l\in\mathscr{Z}^d$. *Then*

$$\operatorname{cond}_2 A^{(d)}(\alpha)=\prod_{k=1}^{d}\frac{\vartheta(q^{2\alpha_{0k}})}{\vartheta(-q^{2\alpha_{0k}})}, \qquad \alpha_0=(\alpha_{01}, ..., \alpha_{0d}), \quad q=e^{-\lambda}.$$

Proof. The symbol function for $A^{(d)}(\alpha)^* A^{(d)}(\alpha)$ is $\{|G_\alpha^{(d)}(\xi)|^2 : \xi\in[-\pi, \pi]^d\}=\{\prod_{k=1}^d |G_{\alpha_k}(\xi_k)|^2 : \xi=(\xi_1, ..., \xi_d)\in[-\pi, \pi]^d\}$. The remainder of the proof is a simple consequence of (2.8). ∎

3. SHIFTED GAUSSIAN CARDINAL INTERPOLATION AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

Let λ be positive and define the Gaussian $\varphi_\lambda(x)=\exp(-\lambda x^2)$, $x\in\mathscr{R}$, and its shift $\varphi_{\lambda,\alpha}=\varphi_\lambda(\cdot+\alpha)$, $\alpha\in\mathscr{R}$; we have changed notation slightly to emphasize dependence on the parameter λ. We have seen that the symbol function $G_\alpha(\cdot, \lambda)$ does not vanish when α is not a half-integer, in which case we can define the *cardinal function* $\chi_{\lambda,\alpha}$ associated with $\varphi_{\lambda,\alpha}$ by its Fourier transform:

$$\widehat{\chi_{\lambda,\alpha}}(\xi):=\frac{\hat{\varphi}_{\lambda,\alpha}(\xi)}{G_\alpha(\xi, \lambda)}, \qquad \xi\in\mathscr{R}, \tag{3.1}$$

where $\hat{\varphi}_{\lambda,\alpha}(\xi)=(\varphi_\lambda(\cdot+\alpha))^\wedge(\xi)$. It is well known that

$$\chi_{\lambda,\alpha}(x)=\sum_{k\in\mathscr{Z}} c_k(\lambda, \alpha)\,\varphi_{\lambda,\alpha}(x-k), \qquad x\in\mathscr{R}, \tag{3.2}$$

where

$$c_k(\lambda, \alpha)=(2\pi)^{-1}\int_{-\pi}^{\pi} G_\alpha(\xi, \lambda)^{-1}\exp(-ik\xi)\,d\xi, \qquad k\in\mathscr{Z}, \tag{3.3}$$

and

$$\chi_{\lambda,\alpha}(j)=\delta_{0j}, \qquad j\in\mathscr{Z}. \tag{3.4}$$

The multivariate cardinal function $\chi_{\lambda,\alpha}^{(d)}$ is defined similarly by its Fourier transform

$$\hat{\chi}_{\lambda,\alpha}^{(d)}(\xi) := \frac{\hat{\varphi}_{\lambda,\alpha}^{(d)}(\xi)}{G_{\alpha}^{(d)}(\xi, \lambda)}, \qquad \xi \in \mathscr{R}^d, \tag{3.5}$$

provided the shift vector α is admissible, that is, none of its components is a half-integer. (Here and elsewhere, $\varphi_{\lambda,\alpha}^{(d)}(x) = \varphi_{\lambda}^{(d)}(x+\alpha)$, where $\varphi_{\lambda}^{(d)}(x) := \exp(-\lambda \|x\|_2^2)$, $x, \alpha \in \mathscr{R}^d$; also $\hat{\varphi}_{\lambda,\alpha}^{(d)}(\xi) = (\varphi^{(d)}(\cdot+\alpha))^\wedge(\xi)$, and $\hat{\chi}_{\lambda,\alpha}^{(d)}(\xi) = (\chi_{\lambda,\alpha}^{(d)}(\cdot))^\wedge(\xi)$.) In this section we shall study some properties of the linear space

$$\operatorname{span}\{\chi_{\lambda,\alpha}^{(d)}(\cdot - k) : k \in \mathscr{Z}^d\}$$

as the parameter λ approaches zero.

The function $\hat{\chi}_{\lambda,\alpha}^{(d)}$ defined in (3.5) inherits the tensor-product structure from the Gaussian and the corresponding symbol. Precisely, we have the useful relation

$$\hat{\chi}_{\lambda,\alpha}^{(d)}(\xi) = \prod_{k=1}^{d} \frac{\hat{\varphi}_{\lambda,\alpha_k}(\xi_k)}{G_{\alpha_k}(\xi_k, \lambda)}, \qquad \xi = (\xi_1, ..., \xi_d), \quad \alpha = (\alpha_1, ..., \alpha_d). \tag{3.6}$$

Primarily because of this tensor-product relation, all of the multivariate results considered in this section can be derived from their univariate counterparts. Therefore we address the univariate topics first.

Poisson summation provides a useful alternative formula for the Fourier transform of the Gaussian cardinal function:

$$\widehat{\chi_{\lambda,\alpha}}(\xi) = (1 + E_{\lambda,\alpha}(\xi))^{-1}, \tag{3.7}$$

where

$$E_{\lambda,\alpha}(\xi) = \sum_{k \in \mathscr{Z} \setminus \{0\}} \frac{\hat{\varphi}_{\lambda,\alpha}(\xi + 2\pi k)}{\hat{\varphi}_{\lambda,\alpha}(\xi)}, \qquad \xi \in \mathscr{R}. \tag{3.8}$$

We shall often infer the behaviour of $\widehat{\chi_{\lambda,\alpha}}$ from that of $E_{\lambda,\alpha}$. It will also be convenient to define $E_\lambda \equiv E_{\lambda,0}$.

LEMMA 3.1. *We have the inqualities*

$$|E_{\lambda,\alpha}(\xi)| \leqslant E_\lambda(\xi) = \sum_{k \in \mathrm{Z} \setminus \{0\}} e^{-((\xi + 2\pi k)^2 - \xi^2)/4\lambda}, \qquad \xi \in \mathscr{R}, \tag{3.9}$$

and

$$E_\lambda(\xi) \leqslant \kappa(\lambda_0) := \sum_{k \in \mathscr{Z} \setminus \{0\}} e^{-\pi(k^2 - |k|)/\lambda_0} \qquad \textit{for} \quad |\xi| \leqslant \pi \textit{ and } \lambda \leqslant \lambda_0. \tag{3.10}$$

Furthermore,

$$\lim_{\lambda \to 0} E_{\lambda, \alpha}(\xi) = 0, \qquad |\xi| < \pi, \tag{3.11}$$

and the convergence is uniform on compact subsets of the interval $(-\pi, \pi)$.

Proof. Inequality (3.9) is a straightforward consequence of the observation $|\hat{\varphi}_{\lambda, \alpha}(\xi)| = |\hat{\varphi}_\lambda(\xi)|$. Moreover, we have the inequalities $\pi k^2 + \xi k \geqslant \pi k^2 - |\xi k| \geqslant \pi(k^2 - |k|)$ for every integer k and every $\xi \in [-\pi, \pi]$. Consequently,

$$E_\lambda(\xi) = \sum_{k \in \mathscr{Z} \setminus \{0\}} e^{-(\pi k^2 + \xi k)/\lambda} \leqslant \sum_{k \in \mathscr{Z} \setminus \{0\}} e^{-\pi(k^2 - |k|)/\lambda} \leqslant \sum_{k \in \mathscr{Z} \setminus \{0\}} e^{-\pi(k^2 - |k|)/\lambda_0},$$

for $\lambda \leqslant \lambda_0$.

Turning to the pointwise convergence, suppose $\xi \in (-\pi, \pi)$ is fixed. Let $\varepsilon > 0$ and choose a sufficiently large integer N so that

$$\sum_{|k| > N} e^{-((\xi + 2\pi k)^2 - \xi^2)/4} \leqslant \varepsilon.$$

Now we can also choose $\lambda_\varepsilon \leqslant 1$ so small that

$$\sum_{|k| \leqslant N} e^{-((\xi + 2\pi k)^2 - \xi^2)/4\lambda} \leqslant \varepsilon, \qquad \lambda \leqslant \lambda_\varepsilon.$$

Thus we have derived the bound

$$E_\lambda(\xi) \leqslant 2\varepsilon, \qquad \lambda \leqslant \lambda_\varepsilon \leqslant 1,$$

and, since $\varepsilon > 0$ was arbitrary, we have established (3.11). The uniform convergence on *compacta* follows from Dini's theorem (see, for example, [H, p. 78]): If we have a monotonic decreasing sequence of continuous real-valued functions on a compact metric space with continuous limit function, then the convergence is uniform. ∎

The asymptotic pointwise behaviour of $\widehat{\chi_{\lambda, \alpha}}$ follows immediately from Lemma 3.1.

THEOREM 3.2. *If* α *is not a half-integer and* $\xi \in (-\pi, \pi)$, *then*

$$\lim_{\lambda \to 0} \widehat{\chi_{\lambda, \alpha}}(\xi + 2\pi j) = \delta_{0j}, \qquad j \in \mathscr{Z}, \tag{3.12}$$

and the convergence is uniform on compact subsets of $(-\pi, \pi)$.

Proof. Equations (3.7) and (3.11) imply (3.12) when $j=0$. When $j\neq 0$, we have

$$|\widehat{\chi_{\lambda,\alpha}}(\xi+2\pi j)| = |\widehat{\chi_{\lambda,\alpha}}(\xi)| \left|\frac{\hat{\varphi}_{\lambda,\alpha}(\xi+2\pi j)}{\hat{\varphi}_{\lambda,\alpha}(\xi)}\right| = |\widehat{\chi_{\lambda,\alpha}}(\xi)|\, e^{-((\xi+2\pi j)^2-\xi^2)/4\lambda},$$

which tends to zero as $\lambda\to 0$ because $\widehat{\chi_{\lambda,\alpha}}(\xi)\to 1$ and $(\xi+2\pi j)^2-\xi^2>0$ for $|\xi|<\pi$ and $j\neq 0$. Uniform convergence on compact subsets of $(-\pi,\pi)$ follows from that of $E_{\lambda,\alpha}(\xi)$ and $e^{-((\xi+2\pi j)^2-\xi^2)/4\lambda}$. ▮

Knowledge of the pointwise behaviour almost everywhere is not sufficient for the integral limits studied below. However, Lemma 3.3, and its consequence Corollary 3.4, will allow us to use the dominated convergence theorem later. Once more the panoply of Theta function theory comes to our aid, in particular the infinite product (2.5).

LEMMA 3.3. *Let $q\leqslant q_0<1$ and let $z=re^{it}$, where $r\in[q,q^{-1}]$ and $|t|<\pi$. Then we obtain the inequalities*

$$\text{(i)}\quad |\vartheta(z,q)|\geqslant T(q_0),\qquad |t|\leqslant \pi/2, \tag{3.13}$$

and

$$\text{(ii)}\quad |\vartheta(z,q)|\geqslant T(q_0)^3\sin^2 t,\qquad \pi/2<|t|<\pi, \tag{3.14}$$

using the notation of Lemma 2.1.

Proof. (i) When $|t|\leqslant\pi/2$, we have $\Re(1+q^{2k+1}r^{\pm 1}\exp(\pm it))\geqslant 1$. Hence every term in the infinite product (2.5) has modulus exceeding one, which implies $|\vartheta(q,z)|\geqslant T(q)$. Furthermore, $T(q)\geqslant T(q_0)$ for $q\leqslant q_0$, yielding (3.13).

(ii) For $k\geqslant 1$ and for any $t\in\mathscr{R}$, the triangle inequality provides

$$|1+q^{2k+1}r^{\pm 1}e^{\pm it}|\geqslant 1-q^{2k}\geqslant 1-q_0^{2k},$$

because $q\leqslant r\leqslant q^{-1}$. Hence

$$|\vartheta(z,q)|\geqslant T(q_0)^3|\,(1+qre^{it})(1+qr^{-1}e^{-it})|.$$

Now $|1+qr^{\pm 1}\exp(\pm it)|\geqslant \inf\{|1+\rho\exp(\pm it)|:\rho\geqslant 0\}=|\sin t|$, the last equation resulting from elementary geometry; this proves (3.14). ▮

The preceding result can be used to derive a upper bound on the Fourier transform of the shifted Gaussian cardinal function. Specifically, if the shift

α is not a half-integer, then the Fourier transform of the Gaussian cardinal function takes the form

$$\widehat{\chi_{\lambda,\alpha}}(\xi) = \frac{\hat{\varphi}_{\lambda,\alpha}(\xi)}{\sum_{k \in \mathscr{Z}} \hat{\varphi}_{\lambda,\alpha}(\xi + 2\pi k)} = \left(\sum_{k \in \mathscr{Z}} e^{2\pi k i \alpha} e^{-(\pi \xi/\lambda) k} e^{-\pi^2 k^2/\lambda} \right)^{-1} = \vartheta(z, q)^{-1}, \tag{3.15}$$

where $q = \exp(-\pi^2/\lambda)$ and $z = q^{\xi/\pi} \exp(2\pi i \alpha)$. Therefore the lower bounds of (3.13) and (3.14) supply upper bounds for the modulus of $\widehat{\chi_{\lambda,\alpha}}$.

Corollary 3.4. *Suppose α is not a half-integer; let $0 < \lambda \leqslant \lambda_0$ and set $q_0 = \exp(-\pi^2/\lambda_0)$, $q = \exp(-\pi^2/\lambda)$. There is a constant $C(\alpha, \lambda_0)$ for which*

$$|\widehat{\chi_{\lambda,\alpha}}(\xi)| \leqslant C(\alpha, \lambda_0), \qquad \lambda \leqslant \lambda_0, \quad |\xi| \leqslant \pi. \tag{3.16}$$

Proof. It suffices to restrict attention to $\alpha \in (-1/2, 1/2)$ because the function $\{\alpha \mapsto |\widehat{\chi_{\lambda,\alpha}}| : \alpha \in \mathscr{R}\}$ is 1-periodic; for $|\alpha| < 1/2$, however, (3.16) follows from (3.13) and (3.14) via (3.15). ∎

Of course, $C(\alpha, \lambda_0) \to \infty$ as $\alpha \to \pm 1/2$.

Armed with these results, let us introduce the family of linear spaces

$$V_{\lambda,\alpha} := \left\{ \sum_{k \in \mathscr{Z}} a_k \chi_{\lambda,\alpha}(\cdot - k) : (a_k)_{k \in \mathscr{Z}} \in l^2(\mathscr{Z}) \right\}, \qquad \lambda > 0. \tag{3.17}$$

The exponential decay of $\chi_{\lambda,\alpha}$ for large argument implies the pointwise convergence of the series for every square-summable sequence. In fact, as will have been immediate to readers familiar with the theory of wavelets, every $V_{\lambda,\alpha}$ is a subspace of $L^2(\mathscr{R})$. More precisely, the integer translates $\{\chi_{\lambda,\alpha}(\cdot - k) : k \in \mathscr{Z}\}$ form a Riesz basis for $L^2(\mathscr{R})$, as we now demonstrate.

Proposition 3.5. *Suppose α is not a half-integer and $0 < \lambda \leqslant \lambda_0$. There exist constants $K_1(\alpha, \lambda_0)$ and $K_2(\alpha, \lambda_0)$ for which*

$$K_1(\alpha, \lambda_0) \sum_{k \in \mathscr{Z}} |a_k|^2 \leqslant \left\| \sum_{k \in \mathscr{Z}} a_k \chi_{\lambda,\alpha}(\cdot - k) \right\|_{L^2(\mathscr{R})}^2$$

$$\leqslant K_2(\alpha, \lambda_0) \sum_{k \in \mathscr{Z}} |a_k|^2, \qquad (a_k)_{k \in \mathscr{Z}} \in l^2(\mathscr{Z}). \tag{3.18}$$

Proof. It suffices to prove (3.18) when the sequence $(a_k)_{k \in \mathscr{Z}}$ is finitely supported, because such sequences form a dense subset of $l^2(\mathscr{Z})$.

Setting $A(\xi) := \sum_{k \in \mathscr{Z}} a_k \exp(-ik\xi)$ and applying the Parseval–Plancherel theorem, we obtain

$$\left\| \sum_{k \in \mathscr{Z}} a_k \chi_{\lambda, \alpha}(\cdot - k) \right\|^2_{L^2(\mathscr{R})} = (2\pi)^{-1} \int_{\mathscr{R}} |A(\xi)|^2 \, |\widehat{\chi_{\lambda, \alpha}}(\xi)|^2 \, d\xi$$

$$= (2\pi)^{-1} \int_{-\pi}^{\pi} |A(\xi)|^2 \sum_{k \in \mathscr{Z}} |\widehat{\chi_{\lambda, \alpha}}(\xi + 2\pi k)|^2 \, d\xi.$$

Now

$$\sum_{k \in \mathscr{Z}} |\widehat{\chi_{\lambda, \alpha}}(\xi + 2\pi k)|^2 = |\widehat{\chi_{\lambda, \alpha}}(\xi)|^2 \left(1 + \sum_{k \in \mathscr{Z} \setminus \{0\}} \left| \frac{\hat{\varphi}_{\lambda, \alpha}(\xi + 2\pi k)}{\hat{\varphi}_{\lambda, \alpha}(\xi)} \right|^2 \right)$$

$$= |\widehat{\chi_{\lambda, \alpha}}(\xi)|^2 \, (1 + E_{\lambda/2}(\xi)). \tag{3.19}$$

Thus (3.10), (3.16) and Parseval's theorem imply

$$\left\| \sum_{k \in \mathscr{Z}} a_k \chi_{\lambda, \alpha}(\cdot - k) \right\|^2_{L^2(\mathscr{R})} \leqslant C(\alpha, \lambda_0)^2 \, (1 + \kappa(\lambda_0/2)) \sum_{k \in \mathscr{Z}} |a_k|^2,$$

for $\lambda \leqslant \lambda_0$. Finally, (3.7), (3.9) and (3.10) provide the inequality

$$|\widehat{\chi_{\lambda, \alpha}}(\xi)|^2 \geqslant (1 + \kappa(\lambda_0))^{-2}, \qquad |\xi| \leqslant \pi, \quad \lambda \leqslant \lambda_0,$$

whence the estimate

$$\left\| \sum_{k \in \mathscr{Z}} a_k \chi_{\lambda,\alpha}(\cdot - k) \right\|^2_{L^2(R)} \geqslant (1 + \kappa(\lambda_0))^{-2} \sum_{k \in \mathscr{Z}} |a_k|^2, \qquad \lambda \leqslant \lambda_0,$$

obtains from (3.19) and Parseval's theorem. ∎

The foregoing result implies that the family of linear maps $\{T_{\lambda, \alpha} : l^2(\mathscr{Z}) \to L^2(\mathscr{R}) : 0 < \lambda \leqslant \lambda_0\}$, where

$$T_{\lambda, \alpha} : (a_k)_{k \in \mathscr{Z}} \mapsto \sum_{k \in \mathscr{Z}} a_k \chi_{\lambda, \alpha}(\cdot - k),$$

is uniformly bounded. It follows that the image of $V_{\lambda, \alpha}$ under the Fourier transform $\mathscr{F}$ is given by the succinct expression

$$\widehat{V_{\lambda, \alpha}} = \widehat{\chi_{\lambda, \alpha}} L^2[-\pi, \pi], \tag{3.20}$$

being the composition $\mathscr{F} \circ T_{\lambda, \alpha}(V_{\lambda, \alpha})$. That is, every member of $\widehat{V_{\lambda, \alpha}}$ is an element of $L^2[-\pi, \pi]$ multiplied by $\widehat{\chi_{\lambda, \alpha}}$. (Here we are using the Riesz–Fischer theorem to pair $l^2(\mathscr{Z})$ and $L^2[-\pi, \pi]$ via the Fourier coefficient sequence.)

Now (3.12) implies the limiting relation

$$\lim_{\lambda \to 0} \widehat{V_{\lambda,\alpha}} = \{\hat{f} \in L^2(\mathscr{R}) : \hat{f} \text{ is supported by } [-\pi, \pi]\} =: \widehat{\mathscr{E}_\pi}. \quad (3.21)$$

We have chosen the notation $\widehat{\mathscr{E}_\pi}$ because its inverse Fourier transform $\mathscr{E}_\pi$ is precisely the set of entire functions of exponential type π; this is the celebrated Paley–Wiener theorem (see, for instance, [SW, pages 108ff]). Thus it is natural to ask whether $V_{\lambda,\alpha} \to \mathscr{E}_\pi$ in some sense, and the answer to this question is particularly elegant in $L^2(\mathscr{R})$. We shall need a specific form of the Poisson summation formula.

LEMMA 3.6. *Let $\hat{f} \in \widehat{\mathscr{E}_\pi}$ and define the continuous function $f \in L^2(\mathscr{R})$ by the inverse Fourier transform*

$$f(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} \hat{f}(\xi)\, e^{ix\xi}\, d\xi, \qquad x \in \mathscr{R}. \quad (3.22)$$

Then we have the equation

$$\sum_{k \in \mathscr{Z}} \hat{f}(\xi + 2\pi k) = \sum_{k \in \mathscr{Z}} f(k) \exp(-ik\xi), \quad (3.23)$$

the second series being convergent in $L^2(\mathscr{R})$.

Proof. See, for instance, [B2, Lemma 3.2]. ∎

As an immediate corollary of Lemma 3.6, we find

$$\sum_{k \in \mathscr{Z}} |f(k)|^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} |\hat{f}(\xi)|^2\, d\xi < \infty. \quad (3.24)$$

Therefore the function

$$I_{\lambda,\alpha} f := \sum_{k \in \mathscr{Z}} f(k)\, \chi_{\lambda,\alpha}(\cdot - k) \quad (3.25)$$

is an element of $V_{\lambda,\alpha}$, by Proposition 3.5, and its Fourier transform is given by

$$\widehat{I_{\lambda,\alpha} f}(\xi) = \widehat{\chi_{\lambda,\alpha}}(\xi) \sum_{k \in \mathscr{Z}} f(k) e^{-ik\xi} = \widehat{\chi_{\lambda,\alpha}}(\xi) \sum_{k \in \mathscr{Z}} \hat{f}(\xi + 2\pi k). \quad (3.26)$$

THEOREM 3.7. *Let $f \in L^2(\mathscr{R})$. If the shift α is not a half-integer, then $\lim_{\lambda \to 0} \operatorname{dist}_2(f, V_{\lambda,\alpha}) = 0$ if and only if $f \in \mathscr{E}_\pi$.*

Proof. Suppose $\hat{f} \in \widehat{\mathscr{E}_\pi}$. We shall prove that $\lim_{\lambda \to 0} \|\hat{f} - \widehat{I_{\lambda,\alpha} f}\|_{L^2(\mathscr{R})} = 0$, which is equivalent to $\lim_{\lambda \to 0} \|f - I_{\lambda,\alpha} f\|_{L^2(\mathscr{R})} = 0$ by the Parseval–Plancherel theorem. Letting I denote the characteristic function of the interval $[-\pi, \pi]$, we have

$$\begin{aligned}
\|\hat{f} - \widehat{I_{\lambda,\alpha} f}\|^2_{L^2(\mathscr{R})} &= \int_{\mathscr{R}} \left| \sum_{k \in \mathscr{Z}} \hat{f}(\xi + 2\pi k) \right|^2 |I(\xi) - \widehat{\chi_{\lambda,\alpha}}(\xi)|^2 \, d\xi \\
&= \int_{-\pi}^{\pi} |\hat{f}(\xi)|^2 \, |1 - \widehat{\chi_{\lambda,\alpha}}(\xi)|^2 \, d\xi \\
&\quad + \int_{-\pi}^{\pi} |\hat{f}(\xi)|^2 \left[\sum_{k \in \mathscr{Z} \setminus \{0\}} |\widehat{\chi_{\lambda,\alpha}}(\xi + 2\pi k)|^2 \right] d\xi \\
&=: I_1 + I_2. \qquad (3.27)
\end{aligned}$$

Now (3.16) implies $|\hat{f}(\xi)|^2 \, |1 - \widehat{\chi_{\lambda,\alpha}}(\xi)|^2 \leqslant |\hat{f}(\xi)|^2 \, (1 + C(\alpha, \lambda_0))^2$ for $|\xi| \leqslant \pi$ and $\lambda \leqslant \lambda_0$. Further, $\hat{f}$ is square integrable on $[-\pi, \pi]$ and $\lim_{\lambda \to 0} |1 - \widehat{\chi_{\lambda,\alpha}}(\xi)|^2 = 0$ by Theorem 3.2. Hence the dominated convergence theorem implies $I_1 \to 0$ as $\lambda \to 0$.

For I_2, Lemma 3.1 provides the bound

$$\begin{aligned}
I_2 &= \int_{-\pi}^{\pi} |\hat{f}(\xi)|^2 \, |\widehat{\chi_{\lambda,\alpha}}(\xi)|^2 \left[\sum_{k \in \mathscr{Z} \setminus \{0\}} \left| \frac{\hat{\varphi}_{\lambda,\alpha}(\xi + 2\pi k)}{\hat{\varphi}_{\lambda,\alpha}(\xi)} \right|^2 \right] d\xi \\
&\leqslant C(\alpha, \lambda_0)^2 \int_{-\pi}^{\pi} |\hat{f}(\xi)|^2 \, E_{\lambda/2}(\xi) \, d\xi.
\end{aligned}$$

Now $E_{\lambda/2}(\xi) \leqslant \kappa(\lambda_0/2)$ for $\lambda \leqslant \lambda_0$ by (3.10), and $\lim_{\lambda \to 0} E_{\lambda/2}(\xi) = 0$ for every $\xi \in (-\pi, \pi)$. Hence a second application of the dominated convergence theorem implies $I_2 \to 0$ as $\lambda \to 0$, and we have shown that $\lim_{\lambda \to 0} \operatorname{dist}_2(f, \mathscr{E}_\pi) = 0$.

Conversely, assume $\lim_{\lambda \to 0} \operatorname{dist}_2(f, V_{\lambda,\alpha}) = 0$ and choose functions $f_\lambda \in V_{\lambda,\alpha}$ for which $\lim_{\lambda \to 0} \|f - f_\lambda\|_{L^2(\mathscr{R})} = 0$. Using (3.20) we obtain the representation

$$\widehat{f_\lambda}(\xi) = \widehat{\chi_{\lambda,\alpha}}(\xi) \, A_\lambda(\xi), \qquad \xi \in \mathscr{R}, \qquad (3.28)$$

where each A_λ is a 2π-periodic function square-integrable on $[-\pi, \pi]$. We shall show that

$$\lim_{\lambda \to 0} \int_{-\pi}^{\pi} |\hat{f}(\xi + 2\pi j)| \, d\xi = 0$$

for every nonzero integer j, which implies $\hat{f} \in \hat{\mathscr{E}}$.

Now (3.28) and the Cauchy–Schwarz inequality provide the relations

$$\begin{aligned}\int_{-\pi}^{\pi} |\widehat{f_\lambda}(\xi+2\pi j)|\, d\xi &= \int_{-\pi}^{\pi} |\widehat{f_\lambda}(\xi)|\, e^{-((\xi+2\pi j)^2-\xi^2)/4\lambda}\, d\xi \\ &\leqslant \|\widehat{f_\lambda}\|_{L^2[-\pi,\pi]} \left(\int_{-\pi}^{\pi} e^{-((\xi+2\pi j)^2-\xi^2)/2\lambda}\, d\xi\right)^{1/2}.\end{aligned}$$

However, $\|\widehat{f_\lambda}\|_{L^2[-\pi,\pi]} \leqslant \|\hat{f}\|_{L^2(\mathscr{R})} + \|\widehat{f_\lambda} - \hat{f}\|_{L^2(\mathscr{R})} = \|\hat{f}\|_{L^2(\mathscr{R})} + o(1)$, as $\lambda \to 0$, whereas direct calculation implies

$$\lim_{\lambda\to 0} \int_{-\pi}^{\pi} e^{-((\xi+2\pi j)^2-\xi^2)/2\lambda}\, d\xi = 0, \qquad j \in \mathscr{Z}\setminus\{0\}.$$

Hence

$$\lim_{\lambda\to 0} \int_{-\pi}^{\pi} |\widehat{f_\lambda}(\xi+2\pi j)|\, d\xi = 0, \qquad j \in \mathscr{Z}\setminus\{0\}.$$

But the triangle inequality and a second application of Cauchy–Schwarz now reveals

$$\begin{aligned}\int_{-\pi}^{\pi} |\hat{f}(\xi+2\pi j)|\, d\xi &\leqslant \int_{-\pi}^{\pi} |\hat{f}(\xi+2\pi j) - \widehat{f_\lambda}(\xi+2\pi j)|\, d\xi \\ &\quad + \int_{-\pi}^{\pi} |\widehat{f_\lambda}(\xi+2\pi j)|\, d\xi \\ &\leqslant (2\pi)^{1/2}\, \|\hat{f} - \widehat{f_\lambda}\|_{L^2(\mathscr{R})} + o(1) = o(1),\end{aligned}$$

as $\lambda \to 0$, which completes the proof. ∎

An almost identical argument yields a result on uniform convergence.

THEOREM 3.8. *Suppose* $\hat{f} \in \widehat{\mathscr{E}_\pi}$. *The functions* $\{I_{\lambda,\alpha} f : \lambda > 0\}$ *converge uniformly to* f *as* $\lambda \to 0$.

Proof. Let g and E_λ be given by (3.23) and (3.9), respectively. We have the relations

$$\begin{aligned}\int_{\mathscr{R}} |\widehat{I_{\lambda,\alpha} f}(\xi)|\, d\xi &= \int_{\mathscr{R}} |\widehat{\chi_{\lambda,\alpha}}(\xi)|\, |g(\xi)|\, d\xi \\ &= \int_{-\pi}^{\pi} \left(\sum_{k\in\mathscr{Z}} |\widehat{\chi_{\lambda,\alpha}}(\xi+2\pi k)|\right) |g(\xi)|\, d\xi. \qquad (3.29)\end{aligned}$$

Since $g \in L^2[-\pi, \pi] \subset L^1[-\pi, \pi]$, and $\sum_{k \in \mathcal{Z}} |\widehat{\chi_{\lambda,\alpha}}(\xi + 2\pi k)| = |\widehat{\chi_{\lambda,\alpha}}(\xi)|$ $(1 + E_\lambda(\xi)) \in L^\infty[-\pi, \pi]$ by (3.16) and (3.10), equation (3.29) implies that $\widehat{I_{\lambda,\alpha} f} \in L^1(\mathcal{R})$. So the Fourier inversion theorem yields the equation

$$f(x) - I_{\lambda,\alpha} f(x) = (2\pi)^{-1} \int_{\mathcal{R}} \left[\sum_{k \in \mathcal{Z}} \hat{f}(\xi + 2\pi k) \right] (I(\xi) - \widehat{\chi_{\lambda,\alpha}}(\xi)) \exp(ix\xi)\, d\xi,$$

where I denotes the characteristic function of the interval $[-\pi, \pi]$. Consequently,

$$\begin{aligned} &|f(x) - I_{\lambda,\alpha} f(x)| \\ &\quad \leqslant (2\pi)^{-1} \int_{-\pi}^{\pi} |\hat{f}(\xi)| \left[\sum_{k \in \mathcal{Z}} |I(\xi + 2\pi k) - \widehat{\chi_{\lambda,\alpha}}(\xi + 2\pi k)| \right] d\xi \\ &\quad = (2\pi)^{-1} \int_{-\pi}^{\pi} |\hat{f}(\xi)|\, |1 - \widehat{\chi_{\lambda,\alpha}}(\xi)|\, d\xi \\ &\qquad + (2\pi)^{-1} \int_{-\pi}^{\pi} |\hat{f}(\xi)| \left[\sum_{k \in \mathcal{Z} \setminus \{0\}} |\widehat{\chi_{\lambda,\alpha}}(\xi + 2\pi k)| \right] d\xi \\ &\quad =: I_1 + I_2, \end{aligned} \tag{3.30}$$

and the similarities between (3.27) and (3.30) are evident.

For I_1, we note that $\lim_{\lambda \to 0} (1 - \widehat{\chi_{\lambda,\alpha}}(\xi)) = 0$ for $|\xi| < \pi$ by Theorem 3.2, and $|1 - \widehat{\chi_{\lambda,\alpha}}(\xi)| \leqslant 1 + C(\alpha, \lambda_0)$ for $\lambda \leqslant \lambda_0$ by (3.16). Moreover, $\hat{f}$ is absolutely integrable on $[-\pi, \pi]$. Thus the dominated convergence theorem allows us to conclude that $\lim_{\lambda \to \infty} I_1 = 0$.

For I_2, we have

$$\sum_{k \in \mathcal{Z} \setminus \{0\}} |\widehat{\chi_{\lambda,\alpha}}(\xi + 2\pi k)| = |\widehat{\chi_{\lambda,\alpha}}(\xi)|\, E_\lambda(\xi), \qquad \xi \in [-\pi, \pi]. \tag{3.31}$$

Now $|\widehat{\chi_{\lambda,\alpha}}(\xi)|\, E_\lambda(\xi)$ is bounded for $\lambda \leqslant \lambda_0$ and $|\xi| \leqslant \pi$ by (3.10) and (3.16). Further, $E_\lambda(\xi)$ converges to zero for $|\xi| < \pi$ by (3.11). Once more the dominated convergence theorem implies $I_2 \to 0$ as $\lambda \to 0$. ∎

One noteworthy consequence of this theorem is the uniform convergence of the shifted Gaussian cardinal functions to the sinc function as λ tends to zero: we simply let f be the sinc function, namely $f(x) = \sin(\pi x)/(\pi x)$.

Proceeding now to be multivariate case, we recall that a vector shift $\alpha = (\alpha_1, ..., \alpha_d) \in \mathcal{R}^d$ is said to be admissible if no α_k is a half-integer. We have seen (cf. (3.6)) that the multivariate cardinal function $\chi_{\lambda,\alpha}^{(d)}$ is simply a tensor product of univariate cardinal functions, that is

$$\chi_{\lambda,\alpha}^{(d)}(x) = \prod_{k=1}^{d} \chi_{\lambda,\alpha_k}(x_k), \qquad x = (x_1, ..., x_d) \in \mathcal{R}^d, \tag{3.32}$$

or, in the Fourier transform domain,

$$\hat{\chi}^{(d)}_{\lambda,\alpha}(\xi) = \prod_{k=1}^{d} \widehat{\chi_{\lambda,\alpha_k}}(\xi_k), \qquad \xi = (\xi_1, ..., \xi_d) \in \mathscr{R}^d. \tag{3.33}$$

These relations imply that the multivariate analogues of our univariate results require rather simple modifications. Therefore we shall only sketch further development.

Equation (3.33) implies the multivariate form of Theorem 3.2:

$$\lim_{\lambda \to 0} \hat{\chi}^{(d)}_{\lambda,\alpha}(\xi + 2\pi j) = \delta_{oj}, \qquad j \in \mathscr{Z}^d, \quad \xi \in (-\pi, \pi)^d, \tag{3.34}$$

the convergence being uniform on compact subsets of $(-\pi, \pi)^d$. Similarly, the shifts $\{\chi^{(d)}_{\lambda,\alpha}(\cdot - k) : k \in \mathscr{Z}^d\}$ form a Riesz basis for $L^2(\mathscr{R}^d)$, so generalizing Lemma 3.11. Thus the linear spaces

$$V^{(d)}_{\lambda,\alpha} = \left\{ \sum_{k \in \mathscr{Z}^d} a_k \chi^{(d)}_{\lambda,\alpha}(\cdot - k) : (a_k)_{k \in \mathscr{Z}^d} \in l^2(\mathscr{Z}^d) \right\}, \qquad \lambda > 0, \tag{3.35}$$

are subspaces of $L^2(\mathscr{R}^d)$ for every admissible shift. Following (3.21), we introduce

$$\hat{\mathscr{E}}^{(d)}_{\pi} := \{ \hat{f} \in L^2(\mathscr{R}^d) : \hat{f} \text{ is supported by } [-\pi, \pi]^d \}, \tag{3.36}$$

and observe that every $f \in \mathscr{E}^{(d)}$ possesses an interpolant

$$I^{(d)}_{\lambda,\alpha} f = \sum_{k \in \mathscr{Z}^d} f(k)\, \chi^{(d)}_{\lambda,\alpha}(\cdot - k)$$

that is a member of $V^{(d)}_{\lambda,\alpha}$. The multivariate incarnations of Theorem 3.7 and Theorem 3.8 are then as follows.

THEOREM 3.9. *Let $f \in L^2(\mathscr{R}^d)$. If $\alpha \in \mathscr{R}^d$ is an admissible shift, then $\lim_{\lambda \to 0} \operatorname{dist}_2(f, V^{(d)}_{\lambda,\alpha}) = 0$ if and only if $f \in \mathscr{E}^{(d)}_{\pi}$.*

THEOREM 3.10. *Suppose $\hat{f} \in \hat{\mathscr{E}}^{(d)}_{\pi}$. The functions $\{I^{(d)}_{\lambda,\alpha} f : \lambda > 0\}$ converge uniformly to f as $\lambda \to 0$.*

4. SHIFTED MULTIQUADRICS

Let c be a non-negative constant. The shifts of the Hardy multiquadric $\varphi_c(x) = (x^2 + c^2)^{1/2}$, $x \in \mathscr{R}$, generate bi-infinite multivariate Toeplitz matrices

$$A(\alpha) := (\varphi_{c,\alpha}(j - k))_{j,k \in \mathscr{Z}}, \qquad \alpha \in \mathscr{R}, \quad \varphi_{c,\alpha}(\cdot) := \varphi_c(\cdot + \alpha), \tag{4.1}$$

which do *not* act as linear operators on $l^2(\mathscr{Z})$. However, it is still possible to analyze their behaviour via the associated tempered distribution symbol function. In particular, it is shown in [B1] that

$$\hat{\varphi}_c(\xi) = -\int_0^\infty e^{-\xi^2/4t}(\pi/t)^{1/2}\, t^{-1}\, d\mu(t), \qquad \xi \in \mathscr{R}\setminus\{0\}, \tag{4.2}$$

where

$$\varphi_c(x) = \varphi_c(0) + \int_0^\infty (1 - e^{-tx^2})\, t^{-1}\, d\mu(t), \qquad x \in \mathscr{R}, \tag{4.3}$$

and it is easily checked that the positive Borel measure μ is given by $d\mu(t) = \exp(-c^2 t)(4\pi t)^{-1/2}\, dt$. Thus the symbol function for $A(\alpha)$ is the sum of tempered distributions

$$\sigma_{c,\alpha}(\xi) = \sum_{k \in \mathscr{Z}} \hat{\varphi}_{c,\alpha}(\xi + 2\pi k) = -\int_0^\infty G_\alpha(\xi, t)\, t^{-1}\, d\mu(t), \tag{4.4}$$

using the Poisson summation formula

$$(\pi/t)^{1/2} \sum_{k \in \mathscr{Z}} e^{-(\xi + 2\pi k)^2/4t} e^{i\alpha(\xi + 2\pi k)} = G_\alpha(\xi, t). \tag{4.5}$$

If α is not a half-integer, then $\Re G_\alpha(\xi, t) > 0$ for every $t \in (0, \infty)$, by Proposition 2.4. Thus $\Re \sigma_{c,\alpha}(\xi) < 0$ for all $\xi \in \mathscr{R}$ and, following [Bu2], we can define the cardinal function by its Fourier transform

$$\hat{\chi}_{c,\alpha}(\xi) = \hat{\varphi}_{c,\alpha}(\xi)/\sigma_{c,\alpha}(\xi), \qquad \xi \in \mathscr{R}, \quad \alpha \notin 1/2 + \mathscr{Z}, \tag{4.6}$$

because the denominator does not vanish.

Formulae (4.2) and (4.3) admit multivariate analogues (see [B1]); specifically,

$$\hat{\varphi}_c^{(d)}(\xi) = -\int_0^\infty e^{-\|\xi\|_2^2/4t}(\pi/t)^{d/2}\, t^{-1}\, d\mu(t), \qquad \xi \in \mathscr{R}^d\setminus\{0\}, \tag{4.7}$$

where the measure $d\mu(t)$ is given as before. Consequently the multivariate symbol $\sigma_{c,\alpha}^{(d)}$ can be expressed as

$$\sigma_{c,\alpha}^{(d)}(\xi) = -\int_0^\infty G_\alpha^{(d)}(\xi, t)\, t^{-1}\, d\mu(t), \qquad \alpha \in \mathscr{R}^d, \tag{4.8}$$

where $G_\alpha^{(d)}$ is the multivariate Gaussian symbol defined in (2.12).

In fact, (4.7) and (4.8) are valid for a large subclass of *conditionally negative definite functions of order one*; see [B1, Theorem 3.6]. However, we prefer to concentrate on the single concrete example of the multiquadric in this study.

Now suppose $\alpha = (\alpha_1, ..., \alpha_d)$ is an inadmissible shift, so that some component, α_{k_0} say, is a half-integer. Then for every $t > 0$, $G_\alpha^{(d)}(\xi, t) = 0$ when ξ_{k_0} an odd integral multiple of π (see Theorem 2.11). Therefore $\sigma_{c,\alpha}^{(d)}(\xi)$ is also zero for such α and ξ, by (4.8). Conversely, if α is an admissible shift and $\xi \in (-\pi, \pi)^d$, then $\Re G_\alpha^{(d)}(\xi, t)$ is no longer positive for all $t > 0$. So, unlike the univariate case, we cannot conclude that $\sigma_{c,\alpha}^{(d)}(\xi) \neq 0$ for such ξ and α. However, it is interesting to note that for a *fixed* $\xi \in (-\pi, \pi)^d$ and an admissible shift α, there exists a $\tilde{c} := \tilde{c}(\xi)$ such that $\sigma_{c,\alpha}^{(d)}(\xi) \neq 0$ for all $c \geq \tilde{c}$. For, we have the relations

$$\sigma_{c,\alpha}^{(d)}(\xi) = \sum_{k \in \mathscr{Z}^d} \hat{\varphi}_{c,\alpha}^{(d)}(\xi + 2\pi k) = \hat{\varphi}_{c,\alpha}^{(d)}(\xi) \left(1 + \sum_{k \in \mathscr{Z}^d \setminus \{0\}} \frac{\hat{\varphi}_{c,\alpha}^{(d)}(\xi + 2\pi k)}{\hat{\varphi}_{c,\alpha}^{(d)}(\xi)}\right) \tag{4.9}$$

and

$$\left|\frac{\hat{\varphi}_{c,\alpha}^{(d)}(\xi + 2\pi k)}{\hat{\varphi}_{c,\alpha}^{(d)}(\xi)}\right| = \left|\frac{\hat{\varphi}_{c,0}^{(d)}(\xi + 2\pi k)}{\hat{\varphi}_{c,0}^{(d)}(\xi)}\right|, \qquad k \in \mathscr{Z}^d. \tag{4.10}$$

Since (see [B2, Equation 2.2)])

$$|\hat{\varphi}_{c,\alpha}^{(d)}(\xi)| = \frac{\pi^{d/2}}{\Gamma(1 + (d/2))} c^{d+1} \int_1^\infty e^{-cs \|\xi\|_2} (s^2 - 1)^{d/2} \, ds > 0, \tag{4.11}$$

and (see [B2, Equation (2.5)])

$$\lim_{c \to \infty} \sum_{k \in \mathbf{Z}^d \setminus \{0\}} \left|\frac{\hat{\varphi}_{c,0}^{(d)}(\xi + 2\pi k)}{\hat{\varphi}_{c,0}^{(d)}(\xi)}\right| = 0, \qquad \xi \in (-\pi, \pi)^d, \tag{4.12}$$

Equation (4.9) provides the estimate

$$|\sigma_{c,\alpha}^{(d)}(\xi)| > 0, \tag{4.13}$$

for large c.

Knowledge of the exact zero structure of the multivariate symbol $\sigma_{c,\alpha}^{(d)}$ eludes the writers at present, and for the duration of the section we shall assume $\alpha = 0$; that is, discussion will be restricted to the unshifted multiquadratic $\varphi_c^{(d)}$.

In Section 3 we studied several convergence properties of Gaussian cardinal interpolants by allowing the parameter λ to tend to zero. Comparable results for multiquadrics can be obtained by allowing the parameter c to tend to infinity, as was first observed in [B2]. The following results supplement the ones already proved therein; the proofs of these results are similar to those in Section 3 and are omitted. However, it is important to understand that relation (4.11) provides the crucial inequality

$$\left|\frac{\hat{\varphi}_{c,\alpha}^{(d)}(\xi)}{\hat{\varphi}_{c,\alpha}^{(d)}(\eta)}\right| \leqslant \exp[-c(\|\xi\| - \|\eta\|)],$$

when $\|\xi\| > \|\eta\| > 0$.

LEMMA 4.1. *Let*

$$\hat{\chi}_c^{(d)}(\xi) := \frac{\hat{\varphi}_c^{(d)}(\xi)}{\sum_{k \in \mathscr{Z}^d} \hat{\varphi}_c^{(d)}(\xi + 2\pi k)}, \qquad \xi \in \mathscr{R}^d.$$

Then

$$\lim_{c \to \infty} \hat{\chi}_c^{(d)}(\xi + 2\pi j) = \delta_{0j}, \qquad \xi \in (-\pi, \pi)^d, \qquad j \in \mathscr{Z}^d,$$

and the convergence is uniform on compact subsets of $(-\pi, \pi)^d$.

THEOREM 4.2. *Let*

$$V_c^{(d)} := \left\{ \sum_{k \in \mathscr{Z}^d} a_k \chi_c^{(d)}(\cdot - k) : (a_k)_{k \in \mathscr{Z}^d} \in l^2(\mathscr{Z}^d) \right\}.$$

Then $V_c^{(d)} \subset L^2(\mathscr{R}^d)$ *for* $c \geqslant c_0$. *Furthermore, if* $f \in L^2(\mathscr{R}^d)$, *then* $\lim_{c \to \infty} \operatorname{dist}_2(f, V_c^{(d)}) = 0$ *if and only if* $\hat{f}$ *is zero almost everywhere outside* $[-\pi, \pi]^d$.

5. CONNECTIONS WITH CARDINAL SPLINES

As indicated in the introductory section, there are several strong semblances between the theory of Gaussian cardinal interpolation (as studied in this paper) and cardinal-spline analysis. These connections will be brought out below.

Suppose $n \geqslant 2$ and let M_n denote the centred cardinal B-spline of order n, *i.e.*, M_n is the n-fold convolution of the characteristic function of the interval $(-1/2, 1/2)$ with itself. Define $\sigma_{n,\alpha}$ to be the shifted B-spline symbol

$$\sigma_{n,\alpha}(\xi) := \sum_{k \in \mathscr{Z}} M_n(k + \alpha) \exp(-ik\xi), \qquad \alpha, \xi \in \mathscr{R}.$$

In complete analogy with Propositions 2.3 and 2.4 of the present paper, we have the following results concerning $\sigma_{n,\alpha}$: For fixed n and α, the function

$\{\xi \mapsto \sigma_{n,\alpha}(\xi) : \xi \in \mathscr{R}\}$ has non-negative real part, and its modulus $|\sigma_{n,\alpha}(\xi)|$ decreases on the interval $0 \leqslant \xi \leqslant \pi$. The first of these results follows quite easily from [JRS, Proposition 3.1], whilst the second was established in [JRS, Theorem 3.2]. Furthermore, Theorem 2.5 of the present paper also holds for the bi-infinite Toeplitz matrix generated by the shifted B-spline ([M] and [BS]); indeed, the zero structure of the shifted B-spline symbol $\sigma_{n,\alpha}(\xi)$ is precisely the same as that of the shifted Gaussian symbol $G_\alpha(\xi, \lambda)$ (compare Theorem 2.5 of this paper with Theorem 2.2 of [S]). As for semicardinal interpolation, our Theorem 2.9 for shifted Gaussians is an exact analogue of the corresponding result for shifted B-splines; the latter may be derived as a consequence of [JRS, Proposition 3.1], via [W, Theorem 5].

It was shown in [deB, Theorem 1] and [JRS, Theorem 3.4] that for fixed ξ and n, the even function $\{\alpha \mapsto |\sigma_{n,\alpha}(\xi)| : \alpha \in \mathscr{R}\}$ decreases on the interval $0 \leqslant \alpha \leqslant 1/2$. We now prove an entirely analogous theorem for G_α. It is interesting to note that, in stark contrast to the B-spline results, the result for the Gaussian (*vide infra*) is a simple extension of our earlier analysis.

PROPOSITION 5.1. *The function* $\{\alpha \mapsto |G_\alpha(\xi, \lambda)| : \alpha \in \mathscr{R}\}$ *is even,* 1-*periodic, and decreases on* $0 \leqslant \alpha \leqslant 1/2$ *for every fixed* $\xi \in \mathscr{R}$ *and* $\lambda > 0$.

Proof. That $|G_\alpha(\xi, \lambda)|$ is an even, 1-periodic function of α is a ready consequence of (2.2). Moreover, the Poisson summation formula implies

$$G_\alpha(\xi, \lambda) = \sum_{k \in ZZ} e^{-\lambda(k+\alpha)^2} e^{-ik\xi} = (\pi/\lambda)^{1/2} \sum_{k \in \mathscr{Z}} e^{-(\xi+2\pi k)^2/(4\lambda)} e^{i\alpha(\xi+2\pi k)}.$$

Setting $\xi =: 2\pi\eta$ and $\beta := 2\pi\alpha$, we obtain

$$\begin{aligned} G_\alpha(\xi, \lambda) &= (\pi/\lambda)^{1/2} e^{-i\beta\eta} \sum_{k \in \mathscr{Z}} e^{-(\pi^2/\lambda)(\eta+k)^2} e^{-ik\beta} \\ &= (\pi/\lambda)^{1/2} e^{-i\beta\eta} G_\eta(\beta, \pi^2/\lambda) \\ &= (\pi/\lambda)^{1/2} e^{-i\beta\eta} e^{-\pi^2\eta/\lambda} \vartheta(e^{-2\pi^2\eta/\lambda} e^{-i\beta}, e^{-\pi^2/\lambda}), \end{aligned}$$

by (2.4), and hence

$$|G_\alpha(\xi, \lambda)| = (\pi/\lambda)^{1/2} e^{-\pi^2\eta/\lambda} |\vartheta(e^{-2\pi^2\eta/\lambda} e^{-i\beta}, e^{-\pi^2/\lambda})|.$$

We have shown in Proposition 2.3 that $\{\beta \mapsto |\vartheta(e^{-2\pi^2\eta/\lambda} e^{-i\beta}, e^{-\pi^2/\lambda})| : 0 \leqslant \beta \leqslant \pi\}$ is a decreasing function for every fixed $\eta \in \mathscr{R}$ and $\lambda > 0$. Therefore the function $\alpha \mapsto |G_\alpha(\xi, \lambda)|$ decreases on $0 \leqslant \alpha \leqslant 1/2$ for every fixed $\xi \in \mathscr{R}$ and $\lambda > 0$. ∎

A prominent theme in the study of univariate cardinal splines has been that of convergence of cardinal spline interpolants as the degree of the underlying spline tends to infinity. Attempts to extend this theory to multivariate splines have led to several interesting results (see, for example, [BHR2, Chapter 5]), including some fascinating connections with problems of tiling [BH]. The studies reported in Sections 3 and 4 of our paper, as well as those carried out in [B2], reveal that the notions of "λ tending to zero" in Gaussian cardinal interpolation and "c tending to infinity" in multiquadric interpolation are natural counterparts of the notions of "degree tending to infinity" in cardinal-spline analysis. As a sample, the reader is invited to compare Theorems 3.13 and 4.2 of the present paper with the main theorem in [BHR1].

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